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RISE OF AN ELIPSOIDAL AIR BUBBLE (A THERMAL) IN THE ATMOSPHERE

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An idealized three-dimensional model of an air bubble (a thermal) rising in the atmosphere is considered. The shape of the thermal is taken to be ellipsoidal. The additional mass, the energy of the circulating motion of the air inside the bubble, the energy of the compensating motion of the air downward in the region of streamlining, and the total energy of the process of rising are determined.

The rise of large air bubbles (thermals) under the effect of the Archimedes force is one of the most common forms of convective motion in the earth's atmosphere. The magnitude of the force depends on the difference of temperatures between the air of the thermal and the surrounding atmosphere at the same height.

One of the possibilities for the appearance of these bubbles is that at the surface of the earth the air is heated due to orographic nonuniformity and, as a consequence, a convective ascending jet is formed. After a time, due to changes in the conditions of heating of the underlying surface or a strong gust the convective jet separates from the earth and a closed air bubble (a thermal) is formed.

It is known that in motion of some bodies in a medium excess pressure appears at the ends of the body and, conversely, a deficit of pressure in the equatorial plane. This gives rise to forces that strive to compress the body in a longitudinal direction and stretch it in a transverse direction. It is obvious that compressive-tensile forces will also act on a moving air bubble. The internal rotational motion of the air in the bubble opposes the forces that strive to deform it [1]. The test data show [2] that the bubble has an oblate shape and indicate the conservation of the axial symmetry of the bubble. The shape of an oblate ellipsoid of rotation that does not strongly differ from a sphere is the closest to the actual shape of a thermal.

In [1] a three-dimensional idealized model of a spherical air bubble is constructed. This is an extreme case of an oblate ellipsoid of rotation. In the present paper the approach of [1] is generalized and an idealized model of an ellipsoidal thermal is constructed. The meridional cross section of this thermal is an ellipse (Fig. 1) with semiaxes a and b and eccentricity $e = c/a = \sqrt{1 - b^2/a^2}$, where c is the half-distance between the foci of the ellipse. The shape of actual thermals [2] does not strongly differ from spherical ($b/a \le 0.85$), so the value of the eccentricity lies within the range $0 < e < 0.5$.

We interpreted the motion of the air bubble as a local process [3]. In this local process (Fig. 2) the core A is the air bubble proper, and the zone of streamlining B is the zone of a descending compensation motion of the surrounding air. In zone B the air is in contact with the core for a comparatively short period of time and virtually retains an initial nonvortical state. Therefore, by analogy with well-known problems of motion of a solid body in liquid [4] we assume a flow in the zone of nonvortical (potential) streamlining. However, on the boundary between the core and the zone of streamlining and also between the layers of the core itself the force of friction acts continuously. Due to this fact, at the initial stage of the process of rising the air of the core acquires and then conserves, along with translatory motion, rotatory motion too, which is characterized by a nonzero angular velocity of the particles of which the core consists. As a result of transfer of angular velocity the rotatory motion occupies the entire volume of the core. Thus, both continuity of physical parameters and force equilibrium of the process as a whole is provided. So, the entire region occupied by the process can be divided into an internal vortex zone A and a region of potential flow B.

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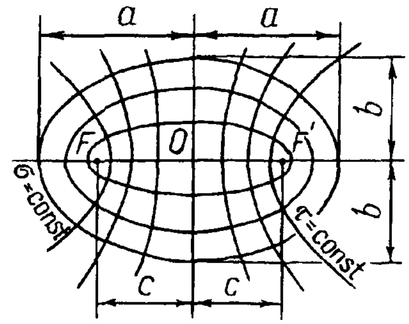

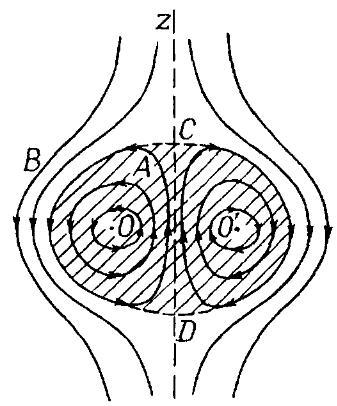

Fig. 1. Coordinates of the oblate ellipsoid of rotation.

Fig. 2. Stream lines in the motion of the ellipsoidal thermal in the atmosphere (OO', central vortex line; A, rising thermal – the core of the process; B, region of streamlining; C and D, front and rear end points; z, vertical axis of symmetry).

The medium is assumed incompressible since the effects of compressibility manifest themselves only at velocities close to the velocity of sound, while the velocity of thermals is of the order of several meters per second.

First, we calculate the flow in the external potential zone – the zone of streamlining. Here we use the ellipsoidal coordinates σ, τ, ε (Fig. 1) [5]. The thermal itself is taken as an initial oblate ellipsoid of rotation. The oblate ellipsoids of rotation (σ = const), one-pole hyperboloids of rotation (τ = const), and half-planes passing through the origin of the coordinates (ε = const) are the coordinate surfaces. The transformation of the ellipsoidal coordinates to the Cartesian rectangular coordinates x, y, z is made by the formulas:

$$x^2 = c^2 (1 + \sigma^2)(1 - \tau^2) \cos^2 \varepsilon , \quad y^2 = c^2 (1 + \sigma^2)(1 - \tau^2) \sin^2 \varepsilon , \quad z = c\sigma\tau . \tag{1}$$

In the ellipsoidal system of coordinates the Lamé coefficients [5] have the form:

$$H_\sigma = c\sqrt{\left(\frac{\sigma^2 + \tau^2}{1 + \sigma^2}\right)} , \quad H_\tau = c\sqrt{\left(\frac{\sigma^2 + \tau^2}{1 - \tau^2}\right)} , \quad H_\varepsilon = c\sqrt{(1 + \sigma^2)(1 - \tau^2)} . \tag{2}$$

The coordinate surface corresponding to $\sigma = \sigma_0$ coincides with the boundary of the initial ellipsoid and thus

$$\sigma_0 = \frac{\sqrt{1 - e^2}}{e} . \tag{3}$$

As is known [6], the velocity of the potentially moving liquid (rot $\vec{v} = 0$) can be presented in the form of the gradient of some scalar function φ – the potential of the velocity: $\vec{v} = \text{grad}\ \varphi$. The conditions of potentiality and noncompressibility of liquid (div $\vec{v} = 0$) determine the Laplace equation for φ

$$\Delta\varphi = 0 ,$$

where Δ = div grad is the Laplace operator.

In the coordinates of the oblate ellipsoid of rotation under the condition of symmetry of the process relative to the vertical axis the Laplace equation has the following form [5]:

$$\frac{\partial}{\partial\sigma}\left[(1 + \sigma^2)\frac{\partial\varphi}{\partial\sigma}\right] + \frac{\partial}{\partial\tau}\left[(1 - \tau^2)\frac{\partial\varphi}{\partial\tau}\right] = 0 . \tag{4}$$

In accordance with the method of separation of the Fourier variables the function $\varphi(\sigma, \tau)$ is presented in the form of the product of two functions each of which involves a dependences on only one variable: $\varphi(\sigma, \tau) =$

$\Omega(\sigma)T(\tau)$. Substituting this expression in (4) and separating the variables, we obtain two second-order ordinary differential equations

$$(1-\tau^2)\frac{d^2T(\tau)}{d\tau^2} - 2\tau\frac{dT(\tau)}{d\tau} + n(n+1)T(\tau) = 0, \tag{5}$$

$$(1+\sigma^2)\frac{d^2\Omega(\sigma)}{d\sigma^2} + 2\sigma\frac{d\Omega(\sigma)}{d\sigma} - n(n+1)\Omega(\sigma) = 0. \tag{6}$$

The first of these equations is the Legendre equation. In turn, the second is reduced to the Legendre equation by complex substitution of $x = \sigma i$, where i is an imaginary unit. The Legendre polynomials [5, 7], by which we, proceeding from (5), (6), constructed the total solution of the initial equation

$$\varphi = cv_0\left[\sum_{n=1}^{\infty} A_n Q_n(\sigma) P_n(\tau) + \sigma\tau\right], \tag{7}$$

are the solutions of Eq. (5). Here v_0 is the velocity of the outer flow in the free space in the transfer coordinate system connected with the thermal; P_n and Q_n are the Legendre polynomials of the n-th power of the first and second order, respectively. To determine the coefficients A_n we use the known relation between the partial derivatives of the stream function and the potential in the curvilinear system of coordinates [4, 7] and then make substitution of (2)

$$\frac{d\psi}{d\tau} = \frac{H_\varepsilon H_\tau}{H_\sigma}\frac{d\varphi}{d\sigma} = c(1+\sigma^2)\frac{d\varphi}{d\sigma}.$$

Substituting expression (7) into the last expression and using (5), we obtain the formula for the stream function which, being equated to zero, gives the equation of the zeroth stream surface

$$\sum_{n=1}^{\infty}\frac{2A_n}{n(n+1)}\frac{dQ_n}{d\sigma}\frac{dP_n}{d\tau} + 1 = 0.$$

From this expression we find the coefficients A_n and, having substituted them into (7), we have the following expression for the potential of the flow past the oblate ellipsoidal thermal:

$$\varphi = -cv_0\left[\frac{\sigma\arctan\sigma - 1}{\arcsin e - e\sqrt{1-e^2}} - \sigma\right]\tau, \tag{8}$$

Proceeding from (8), we can easily obtain the components of the velocity of the flow in the coordinates σ, τ:

$$v_\sigma = -v_0\sqrt{\left(\frac{1+\sigma^2}{\sigma^2+\tau^2}\right)}\left[\frac{\arctan\sigma - \dfrac{\sigma}{1+\sigma^2}}{\arcsin e - e\sqrt{1-e^2}} - 1\right]\tau, \tag{9}$$

$$v_\tau = -v_0\sqrt{\left(\frac{1-\tau^2}{\sigma^2+\tau^2}\right)}\left[\frac{\sigma\arctan\sigma - 1}{\arcsin e - e\sqrt{1-e^2}} - \sigma\right]. \tag{10}$$

On the surface of the initial ellipsoid ($\sigma = \sigma_0$):

$$v_\sigma = 0\,, \quad v_\tau = v_0 \sqrt{\left(\frac{1 - \tau^2}{1 + e^2(\tau^2 - 1)}\right)} \frac{e^3}{\arcsin e - e\sqrt{1 - e^2}}\,. \tag{11}$$

It follows from (11) that the front and rear streamwise end points ($\tau = \pm 1$) are critical – at these points the velocity vanishes. And in the middle (equatorial) plane ($\tau = 0$) the velocity is maximum

$$v_{max} = v_0 \frac{e^3}{\sqrt{1 - e^2}\,(\arcsin e - e\sqrt{1 - e^2})} \tag{12}$$

At values of the eccentricity $0 < e < 0.5$ typical of actual thermals an approximate equality

$$v_{max} = \frac{3}{2}\left[1 + \frac{1}{5}e^2\right] v_0$$

is valid.

For example, the velocity of the outer flow in the middle plane is $v_{max} = 1.503 v_0$ at $e = 0.1$, and $v_{max} = 1.575 v_0$ at $e = 0.5$. Thus, the more oblate is the ellipsoid, the higher the velocity of the streamlining flow in the middle plane.

As is known, accelerated motion of the body in the medium can be considered as occurring in a vacuum, if some additional mass is formally added to the mass of this body [4]. In our case we obtained the following expression for the additional mass of the oblate ellipsoid of rotation:

$$m_{add} = m \frac{\dfrac{e}{\sqrt{1 - e^2}} - \arcsin e}{\arcsin e - e\sqrt{1 - e^2}} \tag{13}$$

Here $m = \rho V$ is the mass of the initial ellipsoid, $V = (4/3)\pi a^3 \sqrt{1 - e^2}$.

When $e < 0.5$, an approximate relation

$$m_{add} = \left[\frac{1}{2} + \frac{3}{10}e^2\right] m\,. \tag{14}$$

is valid.

Thus, the energy of the air in the zone of streamlining is

$$E_p = \left[\frac{1}{2} + \frac{3}{10}e^2\right] E_{kin}\,, \tag{15}$$

where $E_{kin} = m v_0^2/2$.

The distribution of pressure over the surface of the ellipsoid is obtained from the Bernoulli equation

$$p + \rho\frac{v^2}{2} = p_t + E_t\,, \tag{16}$$

where p_t and $E_t = \rho v_0^2/2$ are the pressure and the specific energy of the liquid in the transfer system of reckoning at a large distance from the ellipsoid, respectively; v is the velocity on the surface of the ellipsoid.

From (14) we determine the difference in pressure $\Delta p = p - p_t = E_t - \rho v^2/2$, i.e., the difference of pressures at an arbitrary point on the surface of the thermal and in the free space. Substituting the value of v from (11), we obtain

$$\Delta p = E_t \left[1 - \left(\frac{1 - \tau^2}{1 + e^2(\tau^2 - 1)}\right)\left(\frac{e^3}{\arcsin e - e\sqrt{1 - e^2}}\right)^2\right]. \tag{17}$$

As follows from this formula, the positive difference in pressure acts on the end points of the ellipsoid in the longitudinal direction ($\tau = \pm 1$)

$$\Delta p = E_t > 0 . \tag{18}$$

In the transverse direction ($\tau = 0$)

$$\Delta p = E_t \left[1 - \left(\frac{e^3}{\sqrt{1 - e^2} \arcsin e - e\sqrt{1 - e^2}} \right)^2 \right] .$$

The expression in square brackets is negative at an arbitrary e, consequently

$$\Delta p < 0 . \tag{19}$$

Inequalities (18), (19) show that in the longitudinal direction the ellipsoid is affected by the force which compresses it ($\Delta p > 0$), and in the transverse direction by the tensile force ($\Delta p < 0$). The total estimate of the balance of these forces is made from the value of the integral scalar moment of force [3]

$$M = \int_S (\vec{f} \cdot \vec{r})\, dS , \tag{20}$$

where $\vec{f}$ is the density of the force distributed over the surface S. The integral scalar moment is positive for tensile forces and negative for compressive.

We consider the integral scalar moment of the pressure force. The difference in pressure Δp (17) taken with an opposite sign is the reduced density of the surface force of pressure. The quantity M is divided into longitudinal M_z and transverse M_h components, which in our case have the form

$$M_z = - \int \Delta p r_z \, dS_z , \tag{21}$$

$$M_h = - \overset{8}{\int} \Delta p \, (r_x dS_x + r_y dS_y) . \tag{22}$$

First we find the longitudinal part of the integral scalar moment (21). For this we express r_z in the ellipsoidal coordinates using (1), (3)

$$r_z = z = c\sigma_0 \tau = a\sqrt{1 - e^2}\, \tau . \tag{23}$$

It is known [5] that the component of an arbitrary vector $\vec{F}$, for example, over z in the Cartesian coordinates is determined in terms of the components of this vector in the ellipsoidal coordinates as

$$F_z = \frac{\partial z}{\partial \sigma} \frac{F_\sigma}{H_\sigma} + \frac{\partial z}{\partial \tau} \frac{F_\tau}{H_\tau} + \frac{\partial z}{\partial \varepsilon} \frac{F_\varepsilon}{H_\varepsilon} . \tag{24}$$

We use (24) for the vector of an elementary surface $d\vec{S}$ ($\vec{F} \to d\vec{S}$). The direction of the vector $d\vec{S}$ is taken along the outer normal to the surface of the ellipsoid and, consequently, only the component over σ is not equal to zero ($dS_\tau = dS_\varepsilon = 0$). Thus,

$$dS_z = \left[\frac{\partial z}{\partial \sigma} \frac{dS_\sigma}{H_\sigma} \right]_{\sigma = \sigma_0} = a^2 \tau \, d\varepsilon \, d\tau . \tag{25}$$

The values of all quantities entering (21) are taken for the surface of the ellipsoid, therefore $\sigma = \sigma_0$. Substituting Δp from (17), r_z from (23), and dS_z from (25) into (21), we obtain the expression for M_z

$$M_z = -2\pi a^3 \sqrt{1-e^2}\, E_t \int_{-1}^{+1} \tau^2 \left[1 - \frac{1-\tau^2}{1+e^2(\tau^2-1)} A\right] d\tau\,, \tag{26}$$

where

$$A = \left(\frac{e^3}{\arcsin e - \sqrt{1-e^2}}\right)^2 . \tag{27}$$

Having calculated integral (26), we have

$$M_z = -E_{\text{kin}} \left\{1 + \frac{A}{e^5}\left[3\sqrt{1-e^2}\arcsin e - 3e + e^3\right]\right\}. \tag{28}$$

The expression for a horizontal (22) component of the integral scalar moment is found similarly

$$M_h = E_{\text{kin}} \left\{2 - \frac{3A}{e^5}\left[\frac{\arcsin e}{\sqrt{1-e^2}} - e - \frac{2}{3}e^3\right]\right\}. \tag{29}$$

When $e < 0.5$, using a series of expansions of the functions entering (28), (29), we write expressions (28), (29) approximately in a form convenient for application in practice

$$M_z = -\left(\frac{1}{10} + \frac{9}{350}e^2\right) E_{\text{kin}}\,, \tag{30}$$

$$M_h = \left(\frac{8}{5} + \frac{24}{175}e^2\right) E_{\text{kin}}\,. \tag{31}$$

Thus, it follows from direct calculation that compressive forces ($M_z < 0$) act on the air bubble in the vertical direction and tensile forces ($M_h > 0$) in the horizontal direction. With deviation from the sphere, both compressive and tensile forces acting on the bubble increase and strive to cause further deformation. However, in actuality there is no rapid deformation because the bubble is in force equilibrium with the surrounding atmosphere.

As is shown in [3], in the absence of the source-sink the force equilibrium of the region of the process symmetric relative to some axis is determined by the global condition of equilibrium in the form

$$4E_z - 2E_h = M_h - 2M_z\,, \tag{32}$$

where E_z and E_h are the vertical and horizontal components of the integral kinetic energy of the particles of the air that move inside the bubble, respectively:

$$E_z = \frac{1}{2}\iint_S \rho v_z^2 dS\,, \quad E_h = \frac{1}{2}\iint_S \rho v_h^2 dS\,. \tag{33}$$

To determine the unknown quantities E_z and E_h entering Eq. (32), we need to take into account circulatory motion of the air inside the bubble. Since in this case it moves by the closed surfaces of the stream and the components of the velocity change in a complex manner in transition from one surface of the stream to another, we behave in the following way. We determine the relations between vertical and horizontal components of the velocity and kinetic energy at the peripheral surface of the stream and at the surface of the stream adjacent to the central vortex line OO' (Fig. 2).

The stream lines of the air moving inside the bubble lie in planes passing through the vertical axis of symmetry z. Close to the central vortex line OO' of the vortex tube the stream lines form circles; therefore vertical and horizontal components of kinetic energy for any of these stream lines are equal to each other, consequently, in the central region of the vortex tube $E_z = E_h$.

We now determine the relation between E_z and E_h for surface portions of the bubble. First we find the element of the surface area dS in the ellipsoidal system of coordinates. It is known from a theory of curvilinear coordinates [5] that

$$d\vec{S} = H_\tau H_\varepsilon d\tau d\varepsilon \vec{i_\sigma}, \tag{34}$$

where $\vec{i_\sigma}$ is the unit ort in the direction σ.

Taking account of $|\vec{i_\sigma}| = 1$, and also using formulas (2), we have

$$dS = |d\vec{S}| = H_\tau H_\varepsilon d\tau d\varepsilon = a^2 \sqrt{1 + e^2(\tau^2 - 1)}\, d\tau d\varepsilon\,. \tag{35}$$

Then, from formulas (1), (2), and (24) we obtain the components of the velocity $\vec{v}$ and find the expressions for E_z and E_h. These expressions are reduced to a form valid for the values of the eccentricity $e < 0.5$:

$$E_z = \frac{72}{15}\pi a^2 \left[1 - \frac{3}{5}e^2\right] \rho \frac{v_0^2}{2}, \tag{36}$$

$$E_h = \frac{18}{15}\pi a^2 \left[1 + \frac{2}{5}e^2\right] \rho \frac{v_0^2}{2}. \tag{37}$$

Thus, the ratio of the vertical component of energy E_z to the horizontal component E_h on the surface of the bubble is

$$K = \frac{E_z}{E_h} = 4\,[1 - e^2],$$

and near the central vortex line $K = 1$.

According to equalities (30)-(32)

$$4E_z - E_h = \left[\frac{9}{5} + \frac{33}{175}e^2\right] E_{\text{kin}}\,.$$

Using the limiting values $K = 1$ and $K = 4[1 - e^2]$, we find for the total energy of the motion of the air inside the bubble

$$E_{\text{in}} = E_z + E_h = \left[\frac{9}{5} + \frac{33}{175}e^2\right] E_{\text{kin}}\,,$$

$$E_{\text{in}} = E_z + E_h = \left[\frac{9}{14} + \frac{141}{35}e^2\right] E_{\text{kin}}\,.$$

Combining this value of E_{in} with the energy of the spherical bubble as a whole E_{kin} and the energy of the air in the region of streamlining (15), we come to the final expression for the limits of the total energy of the air involved in the process of the bubble

$$E = E_{\text{kin}} + E_{\text{in}} + E_{\text{p}} = \left[\left(\frac{30}{14} + \frac{1515}{350}e^2\right) \div \left(\frac{33}{10} + \frac{171}{350}e^2\right)\right] E_{\text{kin}}\,. \tag{38}$$

It incorporates the energy of the rise of the bubble as a whole, the energy of the circulation of the air inside the bubble, and the energy in the region of streamlining. The limiting values of the energy E correspond to the limiting values of the coefficient K and indicate the limits within which the value of the integral energy E with a force equilibrium of the thermal is confined.

As follows from (38) the total energy of the air involved in the process of rising of the spherical thermal ($e = 0$) exceeds the energy of the ellipsoidal bubble as a whole E_{kin} from 2.14 to 3.3 times. With increase in the eccentricity this ratio increases still further and at $e = 0.5$ it attains a value from 3.22 to 3.42. The additional mass varies, correspondingly, from (1.14–2.3) m at $e = 0$ to (2.22–2.42) m at $e = 0.5$ and, consequently, a higher value of the additional mass corresponds to a higher value of the eccentricity.

Thus, in the rise of the air bubble the presence of additional mass leads to a decrease of its acceleration and, correspondingly, of the velocity of rise. The additional mass increases still further with compression of the air bubble during its motion, thus leading to further decrease in the velocity of the bubble.

NOTATION

σ, τ, ε, the coordinates of the oblate ellipsoid of rotation; x, y, z, rectangular Cartesian coordinates; $\vec{r}$, radius-vector; r_x, r_y, r_z, the projections of the radius-vector in the rectangular Cartesian coordinates; $\vec{v}$, velocity of the air; v_z, v_h, vertical and horizontal components of the vector of the velocity; c, half-distance between the foci of the ellipse in the meridional cross section of the initial ellipsoid; e, eccentricity of the ellipse; a, large semiaxle of the ellipse; H_σ, H_τ, H_ε, Lamé coefficients; φ, flow potential; ψ, stream function; $\Omega(\sigma)$, $T(\tau)$, functions of separation by the Fourier method; A_n, coefficients; V, volume of the thermal; ρ, density of the air; $\vec{f}$, density of the surface force; M, integral scalar moment; M_z, M_h, vertical and horizontal components of the integral scalar moment; $d\vec{S}$, vector of the elemental area; dS_σ, dS_τ, dS_ε, projections of the vector of the elemental area in the ellipsoidal coordinates; E_{kin}, kinetric energy of the rising bubble as a whole; K, constant equal to the ratio of vertical and horizontal components of the integral kinetic energy; E_{in}, total energy of the motion of the air inside the bubble; E, total energy of the air involved in the process of the rise of the thermal; n, constant introduced in separation of the variables in the Laplace equation. The subscript h indicates transverse components of the quantities.

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